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The Quantum Group as a Symmetry

- The Schrödinger equation of the N -dimensional q -deformed
Harmonic Oscillator -

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Abstract

With the aim to construct a dynamical model with quantum group symmetry, the q -deformed Schrödinger equation of the harmonic oscillator on the N -dimensional quantum Euclidian space is investigated. After reviewing the differential calculus on the q -Euclidian space, the q -analog of the creation-annihilation operator is constructed. It is shown that it produces systematically all eigenfunctions of the Schrödinger equation and eigenvalues. We also present an alternative way to solve the Schrödinger equation which is based on the q -analysis. We represent the Schrödinger equation by the q -difference equation and solve it by using q -polynomials and q -exponential functions. The problem of the involution corresponding to the reality condition is discussed.

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1 Introduction

It is an interesting question whether we can generalize the known theories to a more general framework which may be useful for finding a unified description of the quantum theory and the gravity. It is already a longstanding idea that the spacetime structure has to be modified to achieve this aim, however the question is how this modification has to be. The q -deformation can be understood as a new type of deformation, which is in principle independent of the deformation corresponding to quantum mechanics. Thus the possibility of defining the quantum mechanics on a quantum space is considered as a step in this direction. The quantum space we have here in mind is defined as a q -deformed representation space of the corresponding quantum group, and it is a non-commutative algebra [1, 2]. In this sense, we are considering the quantum mechanics on non-commutative space.

The discovery of the new possibilities related with the q -deformation strongly motivated the physicists to investigate whether the quantum symmetry has some non-trivial impact onto the microscopic spacetime structure [3, 4, 5, 6, 7, 8]. The generalization to the non-commutative geometry is also considered, using the framework of A. Connes (as for the original work see [9]). It is still not clear in which way the approach via the quantum groups relates to his framework. However, with the quantum group symmetry we get rather easily into the non-commutative algebra and can construct for example differential calculi, integrations etc., by using the analogy to the known theory as a guideline.

We take the point of view that the quantum group and the quantum space are the q -deformation of the usual group and space. Then requiring the “ q -correspondence principle”, i.e. in the limit $q \rightarrow 1$ we recover the known undeformed system, we can construct the non-commutative analogue of the known objects such as the quantum Lorentz group [5, 6], the quantum Minkowski space [6, 10, 11, 12, 13], the quantum Poincaré group [14, 15, 16, 11, 17] and many other properties. As for the kinematical aspects the investigations proceeded rather quickly.

As a first step to investigate the dynamics, of course the construction of an appropriate differential calculus is needed. In fact differential calculi on q -spaces have been constructed in a rather early stage of the investigations in quantum groups [3, 4]. However the investigations with respect to the

dynamical features still contain a lot of untouched problems and only a few aspects are investigated so far (For an incomplete list of references see refs.[18-33]).

With the above described motivation the authors have constructed the differential calculus on the N -dimensional q -Euclidian space, i.e. the differential calculus covariant under the action of the quantum group $Fun_q(SO(N))$ ref.[7]. Although it became a little more complicated than the one in refs.[3, 4] which is based on A -type quantum groups, it has the advantage that it contains the metric C_{ij} which makes it possible to define the Laplacian. Using this differential calculus we have investigated the Schrödinger equation corresponding to the q -deformed harmonic oscillator and have computed the ground state energy as well as the first two excited energy levels. However in that stage a systematic construction to all energy levels has been missing.

Recently the investigations on this line were put forward by several authors [19, 20, 21, 22]. Among them, Fiore [21] proposed some kind of ‘raising’ and ‘lowering’ operators which map the wavefunctions of the r -th level to the one of the $(r + 1)$ th level and succeeded to generate all energy levels. However, these operators defined in ref.[21] have an explicit dependence on the energy level, i.e., there is one such operator for each level and thus we have an infinite number of them. Due to this feature, these operators cannot be considered as the q -analogue of the creation-annihilation operator.

Stimulated by the work of ref.[21] we proceeded further our investigations and found the creation-annihilation operator of the N -dimensional q -deformed harmonic oscillator which is level-independent, as one should expect [29]. This defines all energy eigenvalues and gives the expression of the eigenfunctions in terms of creation operators acting on the ground state.

Since the creation-annihilation operator is given in terms of differential operators, in principle all eigenfunctions can be computed. It is however not so easy to find the explicit form of the wave function as a q -polynomial in the coordinates x^i with this method. Thus, we also describe an alternative method to construct the eigenfunctions of the Schrödinger equation, which is another new result of our investigations represented in ref.[29]. This method corresponds to the analytic construction of the wave function in the non-deformed case. Reducing the Schrödinger equation to a q -difference equation we solve it directly by using q -polynomials. The above described two constructions give the same eigenvalues and the resulting wave functions have a one-to-one correspondence.

2 q -Deformed Differential Calculus

Here we review the q -deformed differential calculus on the quantum Euclidean space, i.e., the space on which the quantum group $Fun_q(SO(N))$ is acting, obtained in refs.[7]. As is known, the \hat{R} -matrix of the quantum group $Fun_q(SO(N))$ can be written by using projection operators as

$$\hat{R} = q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{1-N}\mathcal{P}_1, \quad (1)$$

We are considering generic values of q , $0 < q < 1$, therefore the decomposition of the tensor product to the irreducible representations is analogous to the classical case. The projectors \mathcal{P}_S , \mathcal{P}_A and \mathcal{P}_1 give the components of symmetric traceless, antisymmetric and singlet tensors, respectively [2].

The quantum group algebra is characterized by the commutation relation

$$\hat{R}MM = MM\hat{R}, \quad (2)$$

where $M = (M_j^i)$ is a $N \times N$ matrix.

Making use of this projector decomposition we defined the differential calculus by the algebra $\mathbf{C} \langle x^i, dx^i, \partial^i \rangle$ with relations consistent with the quantum group action. The commutation relations of the coordinate functions x^i are given by requiring their q -antisymmetric product to vanish:

$$\mathcal{P}_{A_{kl}}^{ij} x^k x^l = 0. \quad (3)$$

Correspondingly, the wedge product of the one forms on the quantum space is defined by requiring the q -symmetric components to vanish

$$\mathcal{P}_S(dx \wedge dx) = 0, \quad \mathcal{P}_1(dx \wedge dx) = 0. \quad (4)$$

The exterior derivative is defined as a map from the coordinates to the one forms:

$$d : x^i \rightarrow dx^i \quad (5)$$

with the properties of nilpotency:

$$d^2 = 0, \quad (6)$$

the graded Leibniz rule

$$d(ab) = (da)b + (-1)^{ab}a(db), \quad (7)$$

and covariance under the quantum group action. This leads us to the following further relation:

$$x^i dx^j = q \hat{R}_{kl}^{ij} dx^k x^l, \quad (8)$$

To introduce the differential operator ∂^i we postulate the decomposition of the exterior derivative as

$$d = (dx \cdot \partial) = C_{ij} dx^i \partial^j, \quad (9)$$

where C_{ij} is the metric of the quantum space and normalized as

$$C^{ij} C_{ij} = Q_N \equiv \frac{(1 - q^N) \mu}{(1 - q^2)}, \quad (10)$$

C^{ij} is the inverse metric and μ is the factor

$$\mu = 1 + q^{2-N}. \quad (11)$$

With this decomposition and the consistency with other relations, we get the remaining relations:

$$\partial^i x^j = C^{ij} + q \hat{R}_{kl}^{-1ij} x^k \partial^l, \quad (12)$$

$$\mathcal{P}_A{}^{ij}{}_{kl} \partial^k \partial^l = 0, \quad (13)$$

$$\partial^i dx^j = q^{-1} \hat{R}_{kl}^{ij} dx^k \partial^l. \quad (14)$$

This completes the algebra of the differential calculus on N -dimensional q -Euclidian space.

In this algebra, one finds the natural q -analogue of the Laplacian Δ :

$$\Delta = (\partial \cdot \partial) = C_{ij} \partial^i \partial^j. \quad (15)$$

The existence of the Laplacian was one of our motivations to investigate the differential calculus on the q -Euclidian space.

This algebra has another remarkable structure which was found in ref.[18]: First, there is an element Λ which counts the conformal dimensions of the algebra elements as

$$\Lambda x^i = q^2 x^i \Lambda \quad \text{and} \quad \Lambda \partial^i = q^{-2} \partial^i \Lambda \quad (16)$$

where the explicit form of Λ is given by

$$\Lambda = 1 + (q^2 - 1)(x \cdot \partial) + \frac{q^{2-N}(q^2 - 1)^2}{\mu^2}(x \cdot x)\Delta . \quad (17)$$

Then extending the algebra by including Λ^{-1} , one finds an element $\hat{\partial}^i$ with the property:

$$\hat{\partial}^i x^j = C^{ij} + q^{-1} \hat{R}_{kl}^{ij} x^k \hat{\partial}^l . \quad (18)$$

The explicit form of this element is

$$\hat{\partial}^i = \frac{1}{\mu} \Lambda^{-1} [\Delta, x^i] , \quad (19)$$

where $[\cdot, \cdot]$ is the usual commutator. This element satisfies the following simple relations:

$$\hat{\partial}^i \partial^j = q \hat{R}_{kl}^{ij} \partial^k \hat{\partial}^l , \quad (20)$$

$$\mathcal{P}_{Akl}^{ij} \hat{\partial}^k \hat{\partial}^l = 0 . \quad (21)$$

On the other hand, it is known [2] that we can define the $*$ -conjugation by extending the $*$ -structure of the quantum group to a coalgebra antihomomorphism

$$* : \quad x^i, \partial^i \quad \rightarrow \quad (x^i)^* = x_i^*, (\partial^i)^* = \partial_i^* = -q^{-N} \bar{\partial}_i . \quad (22)$$

where $\bar{\partial}_i$ is introduced in such a way that the $*$ -conjugation of eq.(12) becomes simpler. Taking the $*$ -conjugation of the eq.(12) and multiplying \hat{R} on both sides, we obtain

$$\bar{\partial}^i x^{*j} = C^{ij} + q^{-1} \hat{R}_{kl}^{ij} x^{*k} \bar{\partial}^l . \quad (23)$$

Comparing equations (18) and (23), it suggests that one can define an involution for the extended algebra,

$$x^{*i} \equiv (x^j)^* C^{ji} = x^i , \quad (24)$$

$$\bar{\partial}^i \equiv -q^N (\partial^j)^* C^{ji} = \hat{\partial}^i . \quad (25)$$

In ref.[18] the authors proposed to take this as an extension of the reality condition. Note that under this conjugation the invariant “length” $(x \cdot x) = C_{ij} x^i x^j$ and the Laplacian $\Delta = (\partial \cdot \partial)$ transform as

$$(x \cdot x)^* = (x \cdot x) \quad \text{and} \quad (\partial \cdot \partial)^* = q^{-2N} (\bar{\partial} \cdot \bar{\partial}) = q^{-N-2} \Lambda^{-1} \Delta \quad (26)$$

3 The q -Deformed Schrödinger Equation

3.1 The q -deformed harmonic oscillator

The q -deformed Laplacian of the differential calculus on the N -dimensional q -Euclidian space led us to investigate the corresponding Schrödinger equation, the simplest example of which is the harmonic oscillator. The action of its Hamiltonian onto the wave function $|\Psi\rangle$ is defined by

$$H(\omega)|\Psi\rangle = [-q^N(\partial \cdot \partial) + \omega^2(x \cdot x)]|\Psi\rangle = E|\Psi\rangle. \quad (27)$$

We solved this q -deformed Schrödinger equation for the ground state and the first two excited energy levels by using an appropriate ansatz in ref.[7]. The ground state wave function is given by the q -exponential function as

$$|\Psi_0\rangle = \exp_{q^2} \left[\frac{-\omega(x \cdot x)}{q^N \mu} \right]. \quad (28)$$

For the definition of the q -exponential function and some of its properties see the appendix of ref.[CW5]. The ground state energy is

$$H(\omega)|\Psi_0\rangle = E_0|\Psi_0\rangle, \quad \text{where} \quad E_0 = \frac{\omega\mu(1 - q^N)}{(1 - q^2)} = \omega Q_N. \quad (29)$$

For the first excited level we have the eigenfunction of the vector representation $|\Psi_1^i\rangle$:

$$H(\omega)|\Psi_1^i\rangle = E_1|\Psi_1^i\rangle, \quad \text{with} \quad E_1 = \frac{\omega\mu(1 - q^{N+2})}{q(1 - q^2)}, \quad (30)$$

where

$$|\Psi_1^i\rangle = x^i \exp_{q^2} \left[\frac{-\omega(x \cdot x)}{q^{N+1} \mu} \right], \quad (31)$$

and for the second excited level the symmetric tensor $|\Psi_{2,S}\rangle$ and singlet representation $|\Psi_{2,1}\rangle$ with the same energy eigenvalue E_2 :

$$H(\omega)|\Psi_{2,S}^{ij}\rangle = E_2|\Psi_{2,S}^{ij}\rangle, \quad (32)$$

$$H(\omega)|\Psi_{2,1}\rangle = E_2|\Psi_{2,1}\rangle, \quad \text{with} \quad E_2 = \frac{\omega\mu(1 - q^{N+4})}{q^2(1 - q^2)}. \quad (33)$$

The corresponding wave functions are given by

$$|\Psi_{2,S}\rangle = \mathcal{P}_S(x \otimes x) \exp_{q^2} \left[\frac{-\omega(x \cdot x)}{q^{N+2}\mu} \right], \quad (34)$$

and

$$|\Psi_{2,1}\rangle = ((x \cdot x) - \frac{Q_N q^2}{\omega(1+q^2)}) \exp_{q^2} \left[\frac{-\omega(x \cdot x)}{q^{N+2}\mu} \right]. \quad (35)$$

As for the q -numbers we use the conventions:

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (36)$$

and

$$(x)_{q^2} = \frac{1 - q^{2x}}{1 - q^2}. \quad (37)$$

Note that the energy levels of the q -deformed hamornic oscillator are not equidistant and also the factors in the q -exponentials are different for different energy levels.

3.2 The operator formalism

Ordinary quantum mechanics suggests to look for the creation and annihilation operator a^i which satisfies in general a commutation relation with the Hamiltonian of the following form:

$$H(\omega) a^i = q^k a^i [H(\omega) + C(\omega)], \quad (38)$$

where we introduced a possible q -factor q^k (k is a real constant). Such a commutation relation should then generate all eigenfunctions of the q -deformed Schrödinger equation. Then, a^i maps the p th state to the $(p+1)$ th state and the constant $C(\omega)$ gives the energy difference between the two states. However in a q -deformed system there is not such an operator. The reason is that the energy difference between the neighbouring states is not equidistant as we see from the eigenvalues E_0 , E_1 and E_2 . To account for this feature the author of ref.[21] introduced operators separately for each state which raise and lower the energy level, i.e., the p -th raising operator a_p^\dagger acts as : $|p\rangle \rightarrow |p+1\rangle$ (and correspondingly the lowering operator a_p) for the p th

level. In this way he obtains all the states together with an infinite number of raising (and lowering) operators. It is a priori not obvious whether one can define at all a creation-annihilation operator which produces all states of this q -deformed system.

The key point to find the creation-annihilation operator of this system is to allow the quantity $C(\omega)$ to be a function of the Hamiltonian. One can easily see that with this generalization the operator a^i still maps one eigenstate to another eigenstate of different energy level. Using the analogy with the non-deformed case we look for a creation-annihilation operator of the form $(\partial^i + x^i \alpha)$ and the above generalization means that the coefficient α is a function of the Hamiltonian.

In our construction we also have to take into account that the coordinate function x^i and the derivative ∂^i have non-trivial commutation relations with the Hamiltonian:

$$\partial^i H(\omega) = H(q\omega) \partial^i + \mu \omega^2 x^i, \quad (39)$$

$$x^i H(\omega) = q^{-2} H(q\omega) x^i + q^{N-2} \mu \partial^i, \quad (40)$$

With the above described considerations we found the following operators:

Theorem A :

i) The creation operator a_-^i and annihilation operator a_+^i are defined by

$$a_{\pm}^i = q^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} [q^{\frac{N}{2}} \partial^i + x^i \alpha_{\pm}(\omega)], \quad (41)$$

where

$$\alpha_{\pm}(\omega) = \frac{1}{2} [KH(\omega) \pm \sqrt{K^2[H(\omega)]^2 + 4\omega^2}] \quad \text{with} \quad K = \frac{(1-q^2)}{q^{\frac{N}{2}} \mu}, \quad (42)$$

and

$$\lambda^{-\frac{1}{2}} x^i = q^{-\frac{1}{2}} x^i \lambda^{-\frac{1}{2}} \quad \text{and} \quad \lambda^{-\frac{1}{2}} \partial^i = q^{\frac{1}{2}} \partial^i \lambda^{-\frac{1}{2}}. \quad (43)$$

ii) The commutation relation of the creation and annihilation operators with the Hamiltonian is

$$H(\omega) a_{\pm}^i = q^{-1} a_{\pm}^i [H(\omega) - q^{\frac{N}{2}} \mu \alpha_{\pm}(\omega)]. \quad (44)$$

The all proofs of the theorems and propositions given here are published in ref.[29]. Note that the shift operator λ relates to the algebra element Λ mentioned previously as $\lambda^2 = \Lambda$.

With these operators defined in Theorem A we can derive the whole set of states of the corresponding q -deformed Schrödinger equation. We only need to know the eigenvalue of the ground state. As we have shown in the ref.[7], the wavefunction corresponding to the ground state in the limit $q \rightarrow 1$ is given by $|\Psi_0\rangle$ in eq.(28) and it is a candidate of the ground state for the q -deformed case. We can prove that the operator a_+^i annihilates this $|\Psi_0\rangle$:

Proposition A :

$$a_+^i |\Psi_0\rangle = 0 . \quad (45)$$

The proof makes use of the fact that the action of the operator α_\pm onto the ground state is given by

$$\alpha_\pm(\omega) |\Psi_0\rangle = \begin{cases} q^{-\frac{N}{2}} \omega |\Psi_0\rangle & \text{for } \alpha_+ \\ -q^{\frac{N}{2}} \omega |\Psi_0\rangle & \text{for } \alpha_- \end{cases} . \quad (46)$$

Knowing this we act with the annihilation operator a_+^i onto $|\Psi_0\rangle$ and obtain

$$a_+^i |\Psi_0\rangle = q^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} [q^{\frac{N}{2}} \partial^i - q^{-\frac{N}{2}} \omega x^i] |\Psi_0\rangle = 0 . \quad (47)$$

On the other hand, eq.(47) actually defines the ground state wave function.

The excited states are obtained by successively applying the creation operator a_-^i onto the ground state, the energy eigenvalue of which is defined by eq.(29). The energy spectrum can be derived by using eq.(44) which gives the recursion formula of the energy levels from E_p to E_{p+1} when both sides are evaluated on the state $|\Psi_p\rangle$. After some calculation we get the energy eigenvalue of the p -th level as:

Proposition B :

$$E_p = \frac{\omega \mu}{q^{1-N/2}} \left[\frac{N}{2} + p \right] . \quad (48)$$

Another result which is obtained in the course of deriving the energy eigenvalue is the value of the operator α when acting on the state $|\Psi_p\rangle$:

$$\alpha_\pm(\omega) |\Psi_p\rangle = \begin{cases} q^{-p-\frac{N}{2}} \omega |\Psi_p\rangle & \text{for } \alpha_+ \\ -q^{p+\frac{N}{2}} \omega |\Psi_p\rangle & \text{for } \alpha_- \end{cases} . \quad (49)$$

Thus as a consequence, we obtain the equation

$$\begin{aligned} a_{\pm}^i |\Psi_p\rangle &= q^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} [q^{\frac{N}{2}} \partial^i + x^i \alpha_{\pm}(\omega)] |\Psi_p\rangle \\ &= q^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} [q^{\frac{N}{2}} \partial^i \pm x^i q^{\mp(p+\frac{N}{2})} \omega] |\Psi_p\rangle . \end{aligned} \quad (50)$$

Therefore, when the creation-annihilation operator defined in eq.(41) is acting onto an eigenstate and we evaluate only the operator $\alpha_{\pm}(\omega)$ then the resulting expression becomes level-dependent and coincides with the raising operator constructed by Fiore.

One of the important relations to characterize the operators $\alpha_{\pm}(\omega)$ and a_{\pm}^i are the following commutation relations which, in spite of the complicated expression for the operator $\alpha_{\pm}(\omega)$, have a remarkable simple form

Proposition C :

$$\alpha_{\pm}(\omega) a_{+}^i = q^{\pm 1} a_{+}^i \alpha_{\pm}(\omega) , \quad (51)$$

$$\alpha_{\pm}(\omega) a_{-}^i = q^{\mp 1} a_{-}^i \alpha_{\pm}(\omega) , \quad (52)$$

and

$$\lambda^{-\frac{1}{2}} \alpha_{\pm}(\omega) = q \alpha_{\pm}(\omega/q) \lambda^{-\frac{1}{2}} . \quad (53)$$

3.3 Construction of the wave function

With the above operator relations we can easily derive the general form of the wave function for arbitrary energy level. For this end first note that

Theorem B :

The q -antisymmetric product of the creation (annihilation) operator vanishes

$$\mathcal{P}_A(a_{-}^i a_{-}^j) = 0 \quad \text{and} \quad \mathcal{P}_A(a_{+}^i a_{+}^j) = 0 . \quad (54)$$

Theorem B means that the creation operator satisfies the same commutation relation as the q -space coordinate function x^i . Thus for example

$$(a_{-} \cdot a_{-}) a_{-}^i = a_{-}^i (a_{-} \cdot a_{-}) , \quad (55)$$

where $(a_{-} \cdot a_{-}) = C_{ij}(a_{-}^i a_{-}^j)$. Consequently any state constructed by successively applying the creation operator a_{-}^i onto the ground state Ψ_0 is a

q -symmetric tensor. Thus we may call the creation-annihilation operator a_{\pm}^i a q -bosonic operator. Let us state the above results as a theorem.

Theorem C :

The states of the p th level constructed by p creation operators a_-^i

$$|\Psi_p^{i_1 i_2 \dots i_p}\rangle \equiv a_-^{i_1} a_-^{i_2} \dots a_-^{i_p} |\Psi_0\rangle , \quad (56)$$

have the energy eigenvalue $E_p = \omega \mu q^{\frac{N}{2}-1} \llbracket \frac{N}{2} + p \rrbracket$ and are q -symmetric tensors, i.e., $\forall l \in \{1, \dots, p-1\}$

$$\mathcal{P}_{A_{i_l i_{l+1}}}^{jj'} |\Psi_p^{i_1 \dots i_l i_{l+1} \dots i_p}\rangle = 0 . \quad (57)$$

Since the energy eigenvalue depends only on the number of creation operators, the wavefunctions defined in (56) have the same energy eigenvalue E_p for a fixed level p . The second part of Theorem C is a direct consequence of Theorem B.

Thus the number of states of the p th level is equal to the number of states of the non-deformed case, i.e., $\binom{N+p}{p}$. The q -symmetric tensor can be split into symmetric traceless tensors corresponding to the irreducible representations of $Fun_q(SO(N))$:

$$Sym\left(\underbrace{N \otimes N \otimes \dots \otimes N}_p\right) = S_p \oplus S_{p-2} \oplus \dots \oplus \begin{cases} S_1 & \text{for odd } p \\ S_0 & \text{for even } p \end{cases} . \quad (58)$$

Correspondingly the wave function in eq.(56) is split into irreducible components. With this operator method, Theorem C defines in principle all eigenfunctions. However it is not straightforward to obtain an expression of the eigenfunctions as q -polynomials in x^i .

3.4 The wave function as a q -polynomial in x

In order to obtain an expression of the eigenfunctions in terms of q -polynomials in the coordinate, a simple way is to use the relations of the q -differential calculus with the q -analysis [34].

The q -symmetric traceless p th tensor representation S_p^I can be constructed as follows:

$$S_p^I \equiv S_p^{i_1 \dots i_p}(x) = S_{p j_1 \dots j_p}^{i_1 \dots i_p} x^{j_1} \dots x^{j_p} , \quad (59)$$

where

$$\mathcal{P}_{A_{i_j i_{j+1}}}^{kl} S_p^{i_1 \dots i_j i_{j+1} \dots i_p}(x) = 0 , \quad (60)$$

$$C_{i_j i_{j+1}} S_p^{i_1 \dots i_j i_{j+1} \dots i_p}(x) = 0 , \quad (61)$$

for $j = 1, \dots, p-1$.

Since the tensor structure is defined by these q -symmetric tensors, the wave functions can be written by the product of a q -symmetric tensor and a function of x^2 . Thus to solve the Schrödinger equation (27), we take the ansatz:

$$|\Psi\rangle = S_p^I f(x^2) . \quad (62)$$

The problem is to fix the function $f(x^2)$.

For this end we compute the action of the Hamiltonian onto the wave function (62). This lead to the following form of the Schrödinger equation:

$$\begin{aligned} (-q^N \Delta + \omega^2 x^2) S_p^I f(x^2) &= \mu^2 S_p^I [-q^{2N+2p} x^2 D_{x^2}^2 - q^N (\frac{N}{2} + p)_{q^2} D_{x^2} + \frac{\omega^2}{\mu^2} x^2] f(x^2) \\ &= E S_p^I f(x^2) , \end{aligned} \quad (63)$$

where the definition of D_{x^2} is

$$D_{x^2} f(x^2) = \frac{f(q^2 x^2) - f(x^2)}{x^2(q^2 - 1)} . \quad (64)$$

which is the q -difference operator. Therefore eq.(63) is rewritten as a q -difference equation for $f(x^2)$

$$F(D_{x^2}) f(x^2) = [-q^{2N+2p} x^2 D_{x^2}^2 - q^N (\frac{N}{2} + p)_{q^2} D_{x^2} + \frac{\omega^2}{\mu^2} x^2 - \frac{E}{\mu^2}] f(x^2) = 0 . \quad (65)$$

To solve this equation we take an ansatz with the q -exponential function since we already know from our previous considerations that the wave function as to be of such a form.

$$f(x^2) = \sum b_s \text{Exp}_q(q^{-2s} \alpha) \quad \text{where} \quad \alpha = q^{-N-p} \omega . \quad (66)$$

A lengthy but straightforward calculation yields

$$F(D_{x^2})f(x^2) = \frac{q^{\frac{N}{2}-1}\omega}{\mu} \sum \left[-b_{s+1}q^{\frac{N}{2}+p}\llbracket 2s+2 \rrbracket + b_s \left(\llbracket \frac{N}{2} + p + 2s \rrbracket - \frac{E}{q^{\frac{N}{2}-1}\mu\omega} \right) \right] \text{Exp}_q(q^{-2s}\alpha) . \quad (67)$$

Similar to the case of the non-deformed oscillator, we look for the solution which has a finite number of terms in the expansion eq.(66). In such a case the argument of the q -exponential functions in eq.(66) can be shifted to the q -exponential of the largest s . Then such a function $f(x^2)$ becomes a polynomial of x^2 multiplied with the q -exponential which has a smooth finite limit under $q \rightarrow 1$. On the other hand since the q -exponentials with different arguments generate different powers in x^2 , they are independent and cannot cancel each other. Thus solving the equation, we require that for each term in the series eq.(67) the sum of the coefficients of different exponential functions separately vanishes.

First we see that in the term for $\text{Exp}_q(\alpha)$, i.e. for $s = -1$ in eq.(67), the coefficient of b_0 is zero. Therefore we can consistently set $b_s = 0$ for $s < 0$ and require that the first nonzero term starts with the b_0 term.

Now we require that the series has only a finite number of terms. To satisfy this, the second term under the sum, i.e., the coefficient of $\text{Exp}_q(q^{-2s}\alpha)$ containing the factor b_s must vanish for a certain s . Calling this largest integer s as r , this requirement defines the energy eigenvalue E as

$$E = q^{\frac{N}{2}-1}\mu\omega \llbracket \frac{N}{2} + p + 2r \rrbracket . \quad (68)$$

With this eigenvalue, we can set the $b_s = 0$ for $s > r$ and can solve the equation (65). From the condition that the sum of all terms with the same argument in the exponential function vanishes we get the recursion formula

$$b_s = b_{s+1} \frac{-q^{\frac{N}{2}+p}\llbracket 2s+2 \rrbracket}{\llbracket \frac{N}{2} + p + 2r \rrbracket - \llbracket \frac{N}{2} + p + 2s \rrbracket} . \quad (69)$$

Therefore we obtain

$$b_s = b_r \prod_{s \leq t \leq r-1} \frac{q^{\frac{N}{2}+p}\llbracket 2t+2 \rrbracket}{\llbracket \frac{N}{2} + p + 2t \rrbracket - \llbracket \frac{N}{2} + p + 2r \rrbracket} . \quad (70)$$

The b_r simply shows the freedom of the overall normalization and thus the wave functions are now defined in terms of the q -exponential functions and the q -polynomials of the coordinate functions x^i with a finite number of terms and with the b_s given above as

$$|\Psi_{p,r}^I\rangle = S_p^I \sum_{s=0}^r b_s \text{Exp}_q(q^{-2s}\alpha) \quad \text{where} \quad \alpha = q^{-N-p}\omega . \quad (71)$$

The energy eigenvalue of $|\Psi_{p,r}\rangle$ is given by E in eq.(68) and it coincides with the eigenvalue defined by using the creation-annihilation operator in the previous section. We can also confirm that there is a one-to-one correspondence between the wave function given in eq.(56) and the one in eq.(71):

The wave function derived in eq.(71) shows that for each p th rank tensor we have an infinite tower of eigenfunctions labeled by the integer r with the eigenvalue $E_{p+2r} = q^{\frac{N}{2}-1}\mu\omega[\frac{N}{2} + p + 2r]$. This means that for the fixed eigenvalue $E_{p'}$ there is one eigenfunction of p th rank tensor for each p which satisfies $p + 2r = p'$ with a positive integer r . This is the result given in eq.(58).

This completes the q -analytic construction of the eigenfunctions which gives the q -polynomial representation of the wave function corresponding to the irreducible representations of the $Fun_q(SO(N))$.

3.5 Algebra of the creation-annihilation operators

It is interesting to ask how far we can make the analogy of the operator algebra using the creation-annihilation operator. We discuss here some properties of the creation-annihilation operator given in eq.(41).

First we have an alternative representation of the creation-annihilation operator b_{\pm}^i :

Theorem D : The creation-annihilation operator can be also represented by

$$b_{\pm} = \lambda^{-\frac{1}{2}}[q^{\frac{N}{2}}\partial^i + \alpha_{\pm}(q\omega)x^i] , \quad (72)$$

where the operator $\alpha(\omega)$ is given by eq.(42). These two representations satisfy

the identity

$$\frac{1}{\sqrt{K^2[H(\omega)]^2 + 4\omega^2}} b_{\pm} = q^{\frac{3}{2}} a_{\pm} \frac{1}{\sqrt{K^2[H(\omega)]^2 + 4\omega^2}} . \quad (73)$$

In this representation now the operator α_{\pm} is on the left hand side of the coordinate x^i .

The relation between the two representations can be also expressed in the form:

$$b_{\pm} = q^{\frac{3}{2}} [\alpha_+(\omega) - \alpha_-(\omega)] a_{\pm} \frac{1}{[\alpha_+(\omega) - \alpha_-(\omega)]} . \quad (74)$$

The operator b_{\pm}^i is important when we investigate the transformation rule of the creation-annihilation operator under the $*$ -conjugation.

The Hamiltonian can be represented in terms of the creation-annihilation operator using the relations:

$$(a_+ \cdot a_-) = q^{-\frac{3}{2}} \lambda^{-1} \hat{B} [-H(\omega) + Q_N q^{\frac{N}{2}} \alpha_-(\omega)] , \quad (75)$$

$$(a_- \cdot a_+) = q^{-\frac{3}{2}} \lambda^{-1} \hat{B} [-H(\omega) + Q_N q^{\frac{N}{2}} \alpha_+(\omega)] . \quad (76)$$

The linear combination of the above expressions gives the following simple formula

$$(a_+ \cdot a_-) + (a_- \cdot a_+) = -q^{-\frac{3}{2}} (1 + q^N) \lambda^{-1} \hat{B} H(\omega) , \quad (77)$$

where

$$\hat{B} = 1 + \frac{q^2 - 1}{\mu} (x \cdot \partial) . \quad (78)$$

The extra operator $\lambda^{-1} \hat{B}$ commutes with the Hamiltonian:

$$[\lambda^{-1} \hat{B}, H(\omega)] = 0 . \quad (79)$$

Actually the eigenfunctions also form a diagonal basis with respect to this operator, i.e. when acting with the operator $\lambda^{-1} \hat{B}$ onto the wave function eq.(62) we obtain

$$\lambda^{-1} \hat{B} |\Psi_p\rangle = \mathcal{N}_p |\Psi_p\rangle , \quad (80)$$

where \mathcal{N}_p is a certain q -number. However, this eigenvalue \mathcal{N}_p is not independent of the states. By using the explicit form of the wave function

$\Psi_{p+2r} = S_p f_r(x^2)$ derived in section 2 we can determine the eigenvalue \mathcal{N}_p . It is given by

$$\lambda^{-1} \hat{B} |\Psi_{p+2r}\rangle = \frac{q}{\mu q^{\frac{N}{2}}} (q^{-\frac{N}{2}-p+1} + q^{\frac{N}{2}+p-1}) |\Psi_{p+2r}\rangle . \quad (81)$$

From this we see that the eigenvalue of $\lambda^{-1} \hat{B}$ depends only on the tensor structure defined by p and is nonzero. To get the Hamiltonian, we have to divide out this operator.

We also can derive the expression of antisymmetric product of a_+^i and a_-^j :

$$\mathcal{P}_A(a_+^i a_-^j) = \lambda^{-1} q^{\frac{N-3}{2}} \mathcal{P}_A(x^i \partial^j) [\alpha_+(\omega) - q^2 \alpha_-(\omega)] . \quad (82)$$

The operator $\mathcal{P}_A(x^i \partial^j)$ appearing in the r.h.s. is proportional to the angular momentum in the limit $q \rightarrow 1$. The relations shown above suggest that the algebra of creation-annihilation operators closes by including the angular momentum operator. For this point see also ref.[30].

Concerning the Hamiltonian it is not completely straightforward to express it in terms of the creation-annihilation operators as we see from the result of eq.(77). One way to investigate such a property is to take the operators a_\pm^i as the fundamental quantities of the system, and consider the 'improved' Hamiltonian which is directly proportional to $((a_+ \cdot a_-) + (a_- \cdot a_+))$ on operator level. For this one may still consider the rescaling of the creation-annihilation operator by the function of the Hamiltonian as is suggested by the theorem D. From eq.(73), we see that when we define the creation-annihilation operator with the factor $\frac{1}{\sqrt{K^2[H(\omega)]^2 + 4\omega^2}}$ appropriately, the relation between the improved a_\pm^i and b_\pm^i is simplified. Such an improved creation-annihilation operator also seems to simplify the $*$ -conjugation of the operators.

4 Discussion

We have presented the construction of the differential calculus on the q -deformed Euclidian space. There is no problem to introduce the reality condition for the algebra generated only by the coordinate functions x^i . However, the problem of the reality condition for the algebra including the differential

operators is not completely settled. There is a proposal as we explained, for the algebra generated by x^i , ∂^i and Λ^{-1} [18].

We have given two different methods to construct the solution of the q -deformed Schrödinger equation. It is explained how we can obtain the creation-annihilation operator which generates all excitation levels. We also presented the q -analytic method to derive the solutions in terms of the q -polynomial and q -exponential functions by solving the associated q -difference equation.

Still, we have to clarify the conjugation property, since under the conjugation proposed in ref.[18], the Hamiltonian is not an hermitian operator. We considered the various possible Laplacians and its operations on the q -exponential function, and found that $(\partial \cdot \partial)$, $\lambda^{-1}(\bar{\partial} \cdot \partial)$, $\lambda(\partial \cdot \bar{\partial})$ and $(\bar{\partial} \cdot \bar{\partial})$ have simple relations which can be identified with the eigenfunction equation. These four Laplacians are all non-hermitian under the $*$ -conjugation of ref.[18].

On the other hand, the remarkable point is that the Hamiltonian discussed here has real energy eigenvalues. It means that we can solve the equation for $H^*(\omega)$ and we find the eigenfunctions Ψ^* with the same eigenvalue and tensor structure as for the case of $H(\omega)$. There is even a unique correspondence between Ψ and Ψ^* [22]. This properties may be a hint to reconsider the reality condition for the q -deformed case, i.e., rather one should stay in the complexified algebra with trivial $*$ -structure, and impose a reality condition on the observables which can be specified by the q -correspondence principle. A proposal on this point is given in ref.[22]. See also ref.[31].

Finally, let us remark that there is another interesting approach which is independent but possibly related to our approach. There, the quantum space algebra is identified with the complex coordinate of the Bargmann-Fock space [23, 32, 33].

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